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A Contribution to the Problem of L. Fejér on Hermite–Fejér Interpolation

A. K. VARMA

Department of Mathematics, University of Florida, Gainesville, Florida 32611

AND

J. PRASAD

Department of Mathematics, California State University, Los Angeles, California 90032

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1. INTRODUCTION

Let us denote by

$$-1 = x_n < x_{n-1} < \cdots < x_2 < x_1 = +1, \tag{1.1}$$

the n distinct zeros of

$$\pi_n(x) = (1 - x^2)\, P'_{n-1}(x), \tag{1.2}$$

where $P_n(x)$ is the Legendre polynomial of degree n with the normalization

$$P_n(1) = 1. \tag{1.3}$$

Let $f(x)$ be a continuous function defined in $[-1, +1]$. The Hermite interpolation polynomial based on the zeros of (1.2) is given by

$$H_n[f, x] = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} \alpha_k \sigma_k(x), \tag{1.4}$$

where

$$l_k(x) = -\frac{(1 - x^2)\, P'_{n-1}(x)}{n(n-1)(x - x_k)\, P_{n-1}(x_k)}, \qquad k = 1, 2, \ldots, n, \tag{1.5}$$

$$h_k(x) = l_k^2(x), \qquad k = 2, 3, \ldots, n-1, \tag{1.6}$$

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$$h_1(x) = \left[1 - \frac{n(n-1)(1-x)}{2}\right] l_1^2(x), \tag{1.7}$$

$$h_n(x) = \left[1 - \frac{n(n-1)(1+x)}{2}\right] l_n^2(x), \tag{1.8}$$

and

$$\sigma_k(x) = (x - x_k)\, l_k^2(x),\ k = 1, 2,..., n. \tag{1.9}$$

Special cases of $H_n[f, x]$ are

$$R_n[f, x] = \sum_{k=1}^{n} f(x_k)\, h_k(x). \tag{1.10}$$

$$Q_n[f, x] = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} \mu_n'(x_k)\, \sigma_k(x), \tag{1.11}$$

and

$$G_n[f, x] = \sum_{k=1}^{n} f(x_k)\, h_k(x) - \sum_{k=1}^{n} f'(x_k)\, \sigma_k(x). \tag{1.12}$$

Fejér [5, 6] and Grunwald [7] considered these interpolatory polynomials. $R_n[f, x]$, $Q_n[f, x]$, and $G_n[f, x]$ interpolate $f(x)$ at the knots (1.1), but $R_n[f, x]$ has slope zero at all x_k's whereas $Q_n[f, x]$ has slope $\mu_n'(x_k)$ at and x_k $G_n[f, x]$ has there $f'(x_k)$ as slope.

The main object of this paper is to obtain quantitative estimates of $H_n[f, x] - f(x)$ which reflect both the pointwise behavior at the end points as well as the dependence on the smoothness of the function along the lines of the well-known estimates of Steckin [12] for the Fejér operator. Throughout this paper c_1 denotes an absolute positive constant independent of n and x which may vary throughout the text and even within a single statement.

Let $c_\omega[-1, -1]$ be the class of continuous functions on $[-1, -1]$ for which

$$\omega_f(\delta) \leq c\omega(\delta), \tag{1.13}$$

where $\omega_f(\delta)$ is the modulus of continuity of f and $\omega(\delta)$ is a certain modulus of continuity. We now state our main results.

THEOREM 1. *Let* $f(x) \in c_\omega[-1, +1]$. *Then*, for $R_n[f, x]$ *of* (1.10), *we have*

$$|R_n[f, x] - f(x)| \leq \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{(1-x^2)^{1/2}}{n}\right). \tag{1.14}$$

This theorem is an analog to the well-known results of Bojanic [3] and Steckin [12].

Recently, DeVore [4] proved that, for a given continuous function $f(x)$ on $[-1, +1]$, there exists a sequence of polynomials $\mu_n(x) \equiv \mu_n[f, x]$ of degree $\leqslant n$ such that, for $-1 \leqslant x \leqslant +1$,

$$|f(x) - \mu_n(x)| \leqslant c_1 \omega_2 \left(\frac{(1 - x^2)^{1/2}}{n}\right), \tag{1.15}$$

where $\omega_2(\delta)$ is the modulus of smoothness of order 2 of $f(x)$. In our next theorem we prove that there exists a sequence of polynomials $Q_n(f)$ interpolating $f(x)$ at the zeros of (1.2) and satisfying an inequality similar to (1.15).

THEOREM 2. *Let $f \in c[-1, +1]$ and let $\mu_n(x)$ be the sequence of polynomials of DeVore's theorem. Then, for $Q_n[f, x]$ of* (1.11), *we have*

$$|Q_n[f, x] - f(x)| \leqslant c_1 \omega_2 \left(\frac{(1 - x^2)^{1/2}}{n}\right). \tag{1.16}$$

THEOREM 3. *Let $f'(x) \in c[-1, +1]$. Then*, for $G_n[f, x]$ *of* (1.12) *and* $-1 \leqslant x \leqslant +1$, *we have*

$$|G_n[f, x] - f(x)| \leqslant c_1 \frac{(1 - x^2)^{1/2}}{n} \log n \omega\left(f', \frac{1}{n}\right), \tag{1.17}$$

where $\omega(f', \delta)$ is the modulus of continuity of $f'(x)$.

2. PRELIMINARIES

In this section we state some well-known results which we shall use later. Most of these are stated in [1, p. 201]. For the fundamental polynomials $l_k(x)$ we have, for $-1 \leqslant x \leqslant +1$, the following in equalities.

$$l_j^2(x) \leqslant \sum_{k=1}^{n} l_k^2(x) \leqslant 1, \qquad |l_j(x)| \leqslant 1, \qquad j = 1, 2,..., n. \tag{2.1}$$

Equation (2.1) is due to L. Fejér; see [1].

$$(1 - x^2)^{3/4} |P'_{n-1}(x)| \leqslant (2n)^{1/2}, \qquad n \geqslant 4; \tag{2.2}$$

$$(1 - x^2)^{1/2} |P'_{n-1}(x)| \leqslant n - 1. \tag{2.3}$$

Equation (2.2) is due to Bernstein [2]. Equation (2.3) follows from [9, Eq. 2.19, p. 64].

$$|P'_{n-1}(x)| \leqslant \frac{(n-1)(n)}{2}; \tag{2.4}$$

$$\frac{(1-x^2)\,P_n'^2(x)}{n^2} + P_n^2(x) \leqslant 1.$$

For the proof of (2.4), see [11, formula 7.33.8]. It is also known [1, pp. 202–203] that

$$|P_{n-1}(x_k)| \geqslant (8\pi k)^{-1/2}, \qquad k = 2, 3,\ldots, n/2; \tag{2.5}$$

$$|P_{n-1}(x_k)| \geqslant [8\pi(n-1-k)]^{-1/2}, \qquad k = n/2+1,\ldots, n-1; \tag{2.6}$$

$$\sin^2\theta_k = 1 - x_k^2 \geqslant \frac{k^2}{4(n-1)^2}, \qquad k = 2, 3,\ldots, \frac{n}{2}; \tag{2.7}$$

$$\sin^2\theta_k = 1 - x_k^2 \geqslant \frac{(n-1-k)^2}{4(n-1)^2}, \qquad k = \frac{n}{2}+1,\ldots, n-1; \tag{2.8}$$

and

$$\frac{(k-\frac{3}{2})\pi}{n-1} < \theta_k < \frac{k\pi}{n-1}, \qquad k = 2, 3,\ldots, n-1. \tag{2.9}$$

3. Some Lemmas

In this section we prove several lemmas to be used in the proof of Theorems 1–3.

Lemma 3.1. *Let* $x = \cos\theta$ *and*

$$\frac{(j-2)\pi}{n-1} \leqslant \theta \leqslant \frac{j\pi}{n-1}, \qquad j = 2, 3,\ldots, n-1. \tag{3.1}$$

Then

$$|f(x_k) - f(x)| \leqslant \begin{cases} c_1\left[\omega\left(\dfrac{\sin\theta}{n}\right) + \omega\left(\dfrac{1}{n^2}\right)\right], & \text{if } j = k \text{ or } j \pm 1; \\ c_1\left[\omega\left(\dfrac{i\sin\theta}{n}\right) + \omega\left(\dfrac{i^2}{n^2}\right)\right], & \text{if } j < k = j+i \leqslant n \text{ or } 2 \leqslant k = j-i < j. \end{cases} \tag{3.2}$$

The proof depends on the inequalities

$$|x_j - x| \leqslant \frac{4\pi}{n}\sin\theta + \frac{8\pi^2}{n^2}, \qquad |x_k - x| \leqslant \frac{9i\pi}{n}\sin\theta + \frac{41\pi^2 i^2}{n^2}. \tag{3.3}$$

Equation (3.3) can be proved as (2.2) of [8].

LEMMA 3.2. *Let* $-1 \leqslant x \leqslant +1$ *and let* x_j *be that zero of* $P'_{n-1}(x)$ *which is closest to* x. *Then, for* $k = j + i$ *and* $2 \leqslant i \leqslant n - 1$,

$$|l_k(x)| \leqslant c_1/i; \tag{3.4}$$

$$(1 - x_k^2)^{1/4} |l_k(x)| \leqslant \frac{c_1(1 - x^2)^{1/4}}{i}; \tag{3.5}$$

and

$$|l_k(x)| \leqslant \frac{c_1(1 - x^2)^{1/4} n^{1/2}}{i^{3/2}}. \tag{3.6}$$

Proof. By using the definition of x_j, (3.1), and (2.9), it follows that

$$\frac{1}{\sin|(\theta - \theta_k)/2|} \leqslant \frac{4n}{i}, \qquad k \neq j, \qquad k \neq j \pm 1. \tag{3.7}$$

One easily sees that

$$(1 - x_k^2)^{1/2} = \sin \theta_k \leqslant \sin \theta + \sin \theta_k \leqslant 2 \sin \frac{\theta + \theta_k}{2}; \tag{3.8}$$

$$(1 - x^2)^{1/2} = \sin \theta \leqslant \sin \theta + \sin \theta_k \leqslant 2 \sin \frac{\theta + \theta_k}{2}; \tag{3.9}$$

and

$$\frac{1}{\sin((\theta + \theta_k)/2)} \leqslant \frac{1}{\sin|(\theta - \theta_k)/2|}. \tag{3.10}$$

From (2.2), (2.5)–(2.8), (3.7), and (3.8) it follows that

$$\frac{(1 - x^2)^{3/4} |P'_{n-1}(x)|}{n(n-1) \sin|(\theta - \theta_k)/2| \, |P_{n-1}(x_k)| \, (\sin((\theta + \theta_k)/2))^{1/2}} \leqslant \frac{c_i}{i}. \tag{3.11}$$

Now (3.4) follows directly from (1.5), (3.9), and (3.11). Similarly (3.5) follows from (1.5), (3.8), and (3.11). To prove (3.6), we use (1.5), (3.7), (3.10), and (3.11).

LEMMA 3.3. *Let* x_j *be a zero of* P'_{n-1} *closest to* x, $-1 \leqslant x \leqslant +1$. *Then, if* $k = j$ *or* $k = j \pm 1$,

$$(1 - x_k^2)^{1/2} |l_k(x)| \leqslant c_1(1 - x^2)^{1/2}, \tag{3.12}$$

$$|x - x_k| \, l_k^2(x) \leqslant c_1 \frac{(1 - x^2)^{1/2}}{n}. \tag{3.13}$$

Proof. Suppose $k = j$ (the other cases can be proved similarly).

$$(1 - x_k^2)^{1/2} = \sin \theta_k \leqslant \sin \theta + |\sin \theta_k - \sin \theta|$$
$$\leqslant \sin \theta + 2\,|\sin((\theta - \theta_k)/2)|.$$

From (2.5)–(2.8) and (3.8) it follows that

$$|\sin((\theta - \theta_k)/2)|\;|P_{n-1}(x_k)| \geqslant \frac{c_1}{n}, \qquad k = 2, 3,\ldots, n-1. \tag{3.15}$$

Using (1.5), (2.1), (2.3), (3.14), and (3.15), we obtain, for $k = 2, 3,\ldots, n-1$,

$$(1 - x_k^2)^{1/2}\;|l_k(x)| \leqslant (1 - x^2)^{1/2}\;|l_k(x)| + 2\,|\sin((\theta - \theta_k)/2)|\;|l_k(x)|$$
$$\leqslant (1 - x^2)^{1/2} + \frac{(1 - x^2)^{1/2}(1 - x^2)^{1/2}\,|P'_{n-1}(x)|}{n(n-1)\,|\sin((\theta - \theta_k)/2)|\;|P_{n-1}(x_k)|}$$
$$\leqslant c_1(1 - x^2)^{1/2}.$$

This proves (3.12). From (1.5), (2.3) and (2.5),

$$(1 - x)\,l_1^2(x) = \frac{(1 - x)(1 - x^2)\,P'^2_{n-1}(x)}{n^2(n-1)^2} \leqslant \frac{(1 - x^2)^{1/2}}{n}. \tag{3.15a}$$

This proves (3.13) for $k = 1$. Similarly (3.13) holds for $k = n$. For $k = 2, 3,\ldots,$ $n - 1$ we use (3.3), (2.1), (1.5), (2.7), (3.12), and (2.8), and obtain

$$|x - x_k|\;l_k^2(x) \leqslant \left(\frac{4\pi}{n}\sin \theta + \frac{8\pi^2}{n^2}\right) l_k^2(x)$$
$$\leqslant \frac{4\pi}{n}\sin \theta + \frac{8\pi^2}{n^2}\,\frac{c_1(1 - x^2)^{1/2}}{(1 - x_k^2)^{1/2}} \leqslant \frac{c_1 \sin \theta}{n},$$

from which (3.13) follows.

LEMMA 3.4. *If* $-1 \leqslant x \leqslant 1$, *then*

$$\sum_{k=2}^{n-1} (1 - x_k^2)^{1/2}\,l_k^2(x) \leqslant c_1(1 - x^2)^{1/2}, \tag{3.16}$$

$$\sum_{k=1}^{n} |x - x_k|\;l_k^2(x) \leqslant c_1\,\frac{(1 - x^2)^{1/2}}{n}\log n. \tag{3.17}$$

Proof. Equation (3.16) follows from (3.5) and (3.12). Also (3.17) follows from (3.13), (3.3)–(3.6) and (3.15a).

LEMMA 3.5. *Let $h_1(x)$ and $h_n(x)$ be as in* (1.7) and (1.8). Then, for $-1 \leqslant x \leqslant +1$,

$$|f(1) - f(x)|\, h_1(x) \leqslant \frac{1}{n} \sum_{k=1}^{n} \frac{\omega((1 - x^2)^{1/2})}{k}, \tag{3.18}$$

and

$$|f(-1) - f(x)|\, h_n(x) \leqslant \frac{1}{n} \sum_{k=1}^{n} \frac{\omega((1 - x^2)^{1/2})}{k}. \tag{3.19}$$

Proof. For $x = \pm 1$, (3.18) and (3.19) are obvious. For $-1 < x < 1$, it follows from (2.2)–(2.4) that

$$(1 - x)\, h_1(x) \leqslant \frac{c_1(1 - x^2)^{1/2}}{n}, \qquad (1 - x)\, h_n(x) \leqslant \frac{c_1(1 - x^2)^{1/2}}{n}. \tag{3.20}$$

Also, for $-1 < x < 1$,

$$\omega(1 - x) \leqslant \left(1 - \frac{n(1 - x)^{1/2}}{(1 - x)^{1/2}}\right) \frac{\omega((1 - x^2)^{1/2})}{n}. \tag{3.21}$$

Combining (3.20) and (3.21), we obtain

$$|f(1) - f(x)|\, h_1(x) \leqslant c_1 \frac{\omega((1 - x^2)^{1/2})}{n} \leqslant \frac{c_1}{n} \sum_{k=1}^{n} \frac{\omega((1 - x^2)^{1/2})}{k}.$$

This proves (3.18). The proof of (3.19) is similar.

LEMMA 3.6. *For $-1 \leqslant x \leqslant 1$,*

$$\sum_{k=2}^{n-1} \frac{\omega_2((1 - x_k^2)^{1/2})}{n}\, l_k^2(x) \leqslant c_1 \frac{\omega_2((1 - x^2)^{1/2})}{n}. \tag{3.22}$$

Proof. If $x = \pm 1$, this inequality is obvious. For $-1 < x < 1$,

$$\frac{\omega_2((1 - x_k^2)^{1/2})}{n} \leqslant \left(1 + \frac{2(1 - x_k^2)^{1/2}}{(1 - x^2)^{1/2}}\right) \frac{\omega_2((1 - x^2)^{1/2})}{n}.$$

Using (2.1) and (3.16), we obtain

$$\begin{aligned}
&\sum_{k=2}^{n-1} \frac{\omega_2((1 - x_k^2)^{1/2})}{n}\, l_k^2(x) \\
&\quad \leqslant \frac{\omega_2((1 - x^2)^{1/2})}{n} \left[\sum_{k=2}^{n-1} l_k^2(x) + 2 \sum_{k=2}^{n-1} \frac{(1 - x_k^2)^{1/2}}{(1 - x^2)^{1/2}}\, l_k^2(x)\right] \\
&\quad \leqslant \frac{\omega_2((1 - x^2)^{1/2})}{n} [1 + 2c_1] \\
&\quad \leqslant c_1 \frac{\omega_2((1 - x^2)^{1/2})}{n}.
\end{aligned}$$

This proves the lemma.

4. Proof of Theorem 1

From (1.10) and the fact that $R_n[1, x] \equiv 1$, it follows that

$$\begin{aligned} R_n[f, x] - f(x) &= \sum_{k=1}^{n} [f(x_k) - f(x)]\, h_k(x) \\ &= (f(1) - f(x))\, h_1(x) + (f(-1) - f(x))\, h_n(x) \\ &\quad + \sum_{k=2}^{n-1} (f(x_k) - f(x))\, h_k(x) \\ &= F_1(x) + F_2(x) + F_3(x). \end{aligned} \tag{4.1}$$

From Lemma (3.5) we obtain,

$$|F_1(x)| \leqslant \frac{1}{n} \sum_{k=1}^{n} \frac{\omega((1-x^2)^{1/2})}{k}, \qquad |F_2(x)| \leqslant \frac{1}{n} \sum_{k=1}^{n} \frac{\omega((1-x^2)^{1/2})}{k}. \tag{4.2}$$

Using (3.1), we rewrite $F_3(x)$ as

$$\begin{aligned} F_3(x) &= \sum_{k=2}^{n-1} [f(x_k) - f(x)]\, l_k^2(x) \equiv \sum_{k=2}^{n-1} q_k(x) \\ &= q_{j-1}(x) + q_j(x) + q_{j+1}(x) + \sum_{\substack{k\neq j \\ k\neq j\pm 1}} q_k(x). \end{aligned} \tag{4.3}$$

Using (2.1) and Lemmas 3.1, and 3.3 we get

$$\begin{aligned} |q_j(x)| &\leqslant c_1 \left[\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right] l_j^2(x) \\ &\leqslant c_1 \left[\omega\left(\frac{\sin\theta}{n}\right) + \left(1 + \frac{1}{n\sin\theta}\right) \omega\left(\frac{\sin\theta}{n}\right) l_j^2(x)\right] \\ &\leqslant c_1 \omega\left(\frac{\sin\theta}{n}\right)\left[2 + \frac{1}{n}\frac{l_j^2(x)}{\sin\theta}\right] \leqslant c_1 \omega\left(\frac{\sin\theta}{n}\right) \\ &\leqslant \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{\sin\theta}{k}\right). \end{aligned}$$

Hence

$$|q_j(x)| \leqslant \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{\sin\theta}{k}\right). \tag{4.4}$$

Similarly

$$|q_{j-1}(x)| \leqslant \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{\sin\theta}{k}\right), \qquad |q_{j+1}(x)| \leqslant \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{\sin\theta}{k}\right). \tag{4.5}$$

Further, use of Lemma 3.1 enables us to write

$$\sum_{\substack{k\neq j\\ k\neq j\pm 1}} q_k(x) \leqslant c_1 \sum_{\substack{k\neq j\\ k\neq j\pm 1}} \left[\omega\left(\frac{i\sin\theta}{n}\right) + \omega\left(\frac{i^2}{n^2}\right)\right] l_k^2(x)$$

$$\leqslant c_1\left[\sum_{\substack{k\neq j\\ k\neq j\pm 1}} \left\{\omega\left(\frac{i\sin\theta}{n}\right) + \left(1+\frac{i}{n\sin\theta}\right)\omega\left(\frac{i\sin\theta}{n}\right)\right\} l_k^2(x)\right]$$

$$\leqslant c_1\left[\sum_{\substack{k\neq j\\ k\neq j\pm 1}} \omega\left(\frac{i\sin\theta}{n}\right) l_k^2(x) + \sum_{\substack{k\neq j\\ k\neq j\pm 1}} \frac{i}{n}\frac{l_k^2(x)}{\sin\theta}\,\omega\left(\frac{i\sin\theta}{n}\right)\right].$$

Utilizing (3.4) and (3.6) yields

$$\sum_{\substack{k\neq j\\ k\neq j\pm 1}} |q_k(x)| \leqslant c_1 \sum_{\substack{k\neq j\\ k\neq j\pm 1}} \frac{1}{i^2}\,\omega\left(\frac{i\sin\theta}{n}\right). \tag{4.6}$$

Combining (4.1)–(4.4) and (4.6), we obtain

$$|R_n[f,x] - f(x)| \leqslant c_1 \sum_{i=1}^{n} \frac{1}{i^2}\,\omega\left(\frac{i\sin\theta}{n}\right).$$

As in [10], we arrive at:

$$\sum_{i=1}^{n} \frac{1}{i^2}\,\omega\left(\frac{i\sin\theta}{n}\right) \leqslant \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{(1-x^2)^{1/2}}{k}\right). \tag{4.7}$$

Hence

$$|R_n[f,x] - f(x)| \leqslant \frac{c_1}{n} \sum_{k=1}^{n} \omega\left(\frac{(1-x^2)^{1/2}}{k}\right).$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2

We know from [4] that there exists a polynomial $\mu_n[f, x] \equiv \mu_n(x)$ satisfying the inequality

$$|f(x) - \mu_n(x)| \leqslant c_1\omega_2\left(\frac{(1-x^2)^{1/2}}{n}\right). \tag{5.1}$$

Due to uniqueness of Hermite interpolation, we have

$$\mu_n(x) = \sum_{k=1}^{n} \mu_n(x_k)\, h_k(x) + \sum_{k=1}^{n} \mu_n'(x_k)\, \sigma_k(x). \tag{5.2}$$

Using (1.11) and (5.2), we obtain

$$Q_n[f, x] - \mu_n(x) = \sum_{k=1}^{n} [f(x_k) - \mu_n(x_k)]\, h_k(x).$$

But (1.6), (5.1), and Lemma 3.6 give

$$\begin{aligned} Q_n[f, x] - \mu_n(x) &\leqslant c_1 \sum_{k=2}^{n-1} \omega_2 \left(\frac{(1 - x_k^2)^{1/2}}{n}\right) l_k^2(x) \\ &\leqslant c_1 \omega_2 \left(\frac{(1 - x^2)^{1/2}}{n}\right). \end{aligned} \tag{5.3}$$

Combining (5.1) and (5.3), the desired inequality (1.16) follows. This completes the proof of Theorem 2.

6. Proof of Theorem 3

For the proof of this theorem we need the following result of Teljakovski [13, p. 171]. Let $f \in C'[-1, +1]$. Then, for $n \geqslant 9$, there exists a sequence of algebraic polynomials $\phi_n(x)$ of degree $\leqslant n$ such that, for $-1 \leqslant x \leqslant +1$,

$$f(x) - \phi_n(x) \leqslant \frac{c_1(1 - x^2)^{1/2}}{n} \omega \left(f', \frac{(1 - x^2)^{1/2}}{n}\right), \tag{6.1}$$

and

$$f'(x) - \phi_n'(x) \leqslant c_1 \omega \left(f', \frac{(1 - x^2)^{1/2}}{n} + \frac{1}{n^2}\right) \leqslant c_1 \omega \left(f', \frac{1}{n}\right). \tag{6.2}$$

From (1.12), utilizing the fact that $G_n[\phi_n, x] \equiv \phi_n(x)$, we have

$$G_n[f, x] = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} f'(x_k)\, \sigma_k(x) \tag{6.3}$$

and

$$\phi_n(x) = \sum_{k=1}^{n} \phi_n(x_k)\, h_k(x) + \sum_{k=1}^{n} \phi_n'(x_k)\, \sigma_k(x). \tag{6.4}$$

From (6.3) and (6.4),

$$\begin{aligned} G_n[f, x] - \phi_n(x) &= \sum_{k=1}^{n} (f(x_k) - \phi_n(x_k))\, h_k(x) \\ &\quad + \sum_{k=1}^{n} (f'(x_k) - \phi_n'(x_k))\, \sigma_k(x) \\ &= \lambda_1(x) + \lambda_2(x). \end{aligned} \tag{6.5}$$

From (1.6), (3.16), and (6.1) it follows that

$$|\lambda_1(x)| \leqslant \frac{1}{n}\,\omega\left(f', \frac{1}{n}\right) \sum_{k=2}^{n-1} (1 - x_k^2)^{1/2}\, l_k^2(x)$$

$$\leqslant \frac{c_1}{n} (1 - x^2)^{1/2}\, \omega\left(f', \frac{1}{n}\right). \tag{6.6}$$

Next, we estimate $\lambda_2(x)$. For this purpose, we use (6.2) and (3.17), and obtain

$$|\lambda_2(x)| \leqslant c_1 \omega\left(f', \frac{1}{n}\right) \sum_{k=1}^{n} |\sigma_k(x)|$$

$$\leqslant c_1 \frac{(1 - x^2)^{1/2}}{n} \log n\, \omega\left(f', \frac{1}{n}\right). \tag{6.7}$$

From (6.5) and (6.7), we conclude that

$$|G_n[f, x] - \phi_n(x)| \leqslant c_1 \frac{(1 - x^2)^{1/2}}{n} \log n\, \omega\left(f', \frac{1}{n}\right). \tag{6.8}$$

Combining (6.1) and (6.8), we have

$$|G_n[f, x] - f(x)| \leqslant c_1 \frac{(1 - x^2)^{1/2}}{n} \log n\, \omega\left(f', \frac{1}{n}\right).$$

This completes the proof of Theorem 3.

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